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EFFECT OF GRAVITY GRADIENT TORQUE ON THE MOTION OF THE SPIN AXIS OF AN ASYMMETRIC VEHICLE

by C. F. Harding

Prepared under Contract No. NAS 1-4709 by
DOUGLAS AIRCRAFT COMPANY, INC.
Santa Monica, Calif.
for Langley Research Center



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EFFECT OF GRAVITY GRADIENT TORQUE ON THE MOTION OF THE SPIN AXIS OF AN ASYMMETRIC VEHICLE

By C. F. Harding

Research and Development
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SUMMARY

A new method developed by the author was used to determine the effect of gravity gradient torque on the motion of the spin axis of an asymmetric vehicle in a precessing circular orbit about an oblate earth. It was discovered that the relative attitude of two coordinate systems can be expressed as a function of the angular-velocity time history as an alternate for the usual parameters. When Euler angles are employed, a highly nonlinear system of six coupled equations results. The system can be reduced to a single vector equation for the unknown angular velocity, ω . The equation is particularly simple and readily lends itself to a complete formal solution by the theory of iterations. An acceptable approximation of ω leads to the summation of an infinite series for the determination of instantaneous attitude. This general procedure was employed to investigate the stability of a spin direction in space. The parameters of the Manned Orbiting Research Laboratory (MORL) were used. The investigation shows there is no appreciable coning; specifically, the spin axis (axis of maximum moment of inertia) deviates no more than 0.4 seconds of arc from the angular momentum vector as it wanders in space. There is, however, a larger, yet slow, variation of angular momentum from initial position, although this is tolerable because it is no more than 2.5° . An important discovery was that the upper bound of 2.5° is independent of boundary conditions which involve different spin-axis angles to the Earth's equator. An extension of the method to elliptic orbits is outlined in the appendix.

INTRODUCTION

The creation of on-board electrical power is an important requirement of a manned orbiting space laboratory. If solar-cell arrays are used for this purpose, they must be sun-oriented continuously for long periods of time. To maintain the desired orientation, the space vehicle will be spin-stabilized along a sun-vehicle axis. However, as the Earth advances along its orbit about the sun, the space vehicle solar cells will become misaligned.

An on-board system capable of realigning the vehicle will be periodically activated to correct such misalignment. Further, the vehicle will be subject continuously to Earth gravity gradients, which will cause the spin axis to precess. Spin-axis precession greater than 30° from the sun-pointing direction could cause a power loss of approximately 15%.

For the analysis, two assumptions are made: (1) the vehicle is a perfectly rigid asymmetric body (no energy loss caused by rotation), and (2) the effect of an oblate Earth is accounted for by a precessing circular orbit (no field change caused by oblateness). The first assumption is necessary to formulate the problem; the second assumption is known to be accurate.

The problem of gravity gradient torque has been studied by many writers in recent years. The usual procedure used to obtain an analytical solution as a function of time is to linearize the equations of motion and to consider the body as having an axis of symmetry about which it spins. Because linearization is inadequate for the problem considered here, a more general approach is used. The author discovered that the problem can be completely solved formally by critically reviewing the concept of attitude. It was determined that attitude is a simple function of the angular-velocity history in infinite series. With this description, a simple vector equation that lends itself to iterations can be obtained. The convergence is fast. A detailed exposition of the method follows.

SYMBOLS

| | |
|------------------|---|
| $A(t)$ | matrix-dyadic taking B^* into B |
| A | semimajor axis of elliptic orbit |
| B | frame attached to vehicle |
| B^* | frame irrotational in space |
| E | idemfactor or 3×3 unit matrix |
| e | eccentricity |
| $\hat{e}_i(t)$ | unit vectors of B^* , seen in B |
| $\hat{e}_i^*(t)$ | unit vectors of B^* , seen in B^* |
| G | gravitational constant for the Earth |
| $\Gamma(t)$ | (see equation (8)) |
| $\Gamma^{-1}(t)$ | (see equation (8)) |
| I | moment of inertia dyadic at center of mass, seen in B |
| i | orbit inclination |
| λ | ratio of G to R^3 , or $3q^2$ |
| n | mean angular motion in orbit |
| ν | true anomaly |
| o | null or zero vector |
| p | orbit precession rate |
| ϕ | initial vehicle spin azimuth |
| $\phi_n(t)$ | (see equation (7)) |
| q | orbit angular velocity |
| R | distance from Earth center to vehicle |
| $\hat{r}(t)$ | Earth-to-vehicle direction, seen in B |
| $\hat{r}^*(t)$ | Earth-to-vehicle direction, seen in B^* |
| $\hat{r}_n(t)$ | partitioning of $\hat{r}(t)$ in terms of λ^n |

| | |
|---------------|---|
| s | initial vehicle spin along No. 3 axis |
| t | time |
| θ | initial vehicle spin-noding angle |
| $\oplus(t)$ | (see equation (7)) |
| $\omega(t)$ | angular velocity of B about B*, seen in B |
| $\omega_n(t)$ | partitioning of $\omega(t)$ in terms of λ^n |

GENERAL THEORY OF RIGID BODY MOTION

Introduction

The interaction of three point masses under an inverse square law and the rotation of a heavy asymmetric top about a fixed point are the two most outstanding unsolved problems in Classical Mechanics. The solutions for these problems are unknown in what is loosely called "closed form," which usually means the solution is specified or described by means of familiar functions and/or a finite set of integrations of known expressions (quadratures). The language employed is arbitrary, since even the simplest transcendental function, e^t , is ultimately defined in terms of an infinite process and so is not strictly closed. The general second-order linear differential equation with time-varying coefficients, furthermore, has been proved unsolvable in closed form. The hope of solving all physical problems in closed form is thus doomed at the start. The best alternative is to relax the restrictions of closed form to what is called a formal integration, in which each step toward the solution is specified in complete detail as a definite set of algebraic operations and/or quadratures, whether or not their number is finite. Thus, the definition of a formal integration allows the steps to be carried out in principle, such as in the expression for e^t . The goal in this section is the formulation of a method by which a large class of rigid body problems can be formally solved.

The motion of a rigid body about its center of mass is described by Euler's dynamical equations in connection with some kinematic relationships between the attitude variables and the angular-velocity components. Euler's equations, which merely state that the time rate of change of angular momentum is equal to the applied torque, are written with respect to the body itself. The equations form a set of three coupled nonlinear first-order differential equations for the angular-velocity components ω_i ($i = 1, 2, 3$). The equations are

$$I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = \ell_1$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 = \ell_2$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = \ell_3$$

The torque components appear as forcing functions relative to the homogeneous equations (the free state) and usually are dependent on the attitude variables and the time. The kinematic relations are of three types, depending upon whether direction cosines, Euler angles, or a set of four

parameters are employed. The direction cosine specification is composed of nine first-order linear differential equations with six algebraic constraints. The four-parameter set satisfies four first-order linear differential equations with one algebraic constraint. On the other hand, Euler angles (the most commonly used) satisfy three highly nonlinear first-order differential equations. Euler angles are used because relatively simple equations result when small motion approximations are introduced. The exact analysis, however, is complex and requires a simultaneous solution of 12, 7, or 6 coupled nonlinear equations, depending on the parameterization of attitude. The method to be presented introduces a new description of attitude which reduces the number of dependent variables to three--the angular-velocity components. The mathematics is carried out exclusively in vector-dyadic terminology, for which this problem is especially suited. This compact formulation takes the form of a simple nonlinear first-order vector integro-differential equation. The formal integration is then carried out in two steps: (1) an iterative procedure to obtain $\omega(t)$, and (2) substitution of $\omega(t)$ into the kinematics. By this approach, the original equation is shown to be equivalent to an infinite set of linear first-order vector differential equations. The set is not simultaneous, however, but orders itself sequentially in a series of better approximations to the exact value, that is, iteration. The theory of iterations is a tool of analysis for formally solving a restricted class of differential equation systems. But the majority of equations is not amenable to such an approach to a formal solution, because, in general, all the steps are not explicit. Most importantly, the equations for the higher approximations cannot be written in detail and thus cannot be solved. The new method of expressing attitude accomplishes a transformation of the mathematics to an equation that can be handled by iterations. The process is started by solving the free-body equation in terms of elliptic functions to get $\omega_0(t)$. The time variation of $\omega_0(t)$ is then substituted into the equation for $\omega_1(t)$, which is inverted to give $\omega_1(t)$ as a definite quadrature. These steps are repeated with $\omega_2(t)$ expressed as a quadrature involving the known $\omega_0(t)$, $\omega_1(t)$, and so on, to infinity. Thus,

$$\omega(t) = \omega_0(t) + \lambda \omega_1(t) + \lambda^2 \omega_2(t) + \dots$$

The method is convergent for small enough λ , where λ is a measure of the torque strength if, initially, the body is spinning near a stable axis. The stable axes for an asymmetric body are the maximum and minimum moment of inertia axes. The intermediate axis is unstable. The angular velocity, obtained from a well-defined set of quadratures, is then, by definition, a first requirement for the formal solution. Finally, the instantaneous attitude must be expressed. This is done by using $\omega(t)$ in a special coordinate transformation that is a function of the $\omega(t)$ history. This calculation of attitude is simply a last quadrature to be performed; hence, the complete problem has been formally solved in the mathematical sense. The analysis, while basically theoretical, is practical because the execution of each successive approximation involves only time and patience. It is especially convenient to apply the new method to gravity gradient effects because the results can be expressed entirely in terms of circular functions. Only the

first perturbation must be calculated, as a result of both the smallness of the torque and the high initial spin. The next three sections present a new formulation of attitude, its application to gravity gradient torque in circular orbit, and the formal solution.

Kinematic Relations

Notation convention. — Two co-origin rectangular cartesian frames are assumed: (1) B^* , which is nonrotational relative to space, and (2) B , which is attached to the vehicle, both at the center of mass. The following convention will be used: if v_i^* and v_i ($i = 1, 2, 3$) are the components expressed in B^* and B , respectively, of a certain vector, then two vectors, \mathbf{v}^* and \mathbf{v} , are constructed in a third frame, B^\dagger , with components of v_i^* and v_i . Note that \mathbf{v}^* is carried into \mathbf{v} by the operation of a dyadic, A , whose components in B^\dagger are equal to the elements of the transformation matrix that takes the v_i^* into the v_i . Although not standard, the notation makes the following analysis easier to handle.

Attitude as rotational history. — Let $\hat{\epsilon}_i^*$ and $\hat{\epsilon}_i$ ($i = 1, 2, 3$) be the unit vectors of the frame B^* . Note that the $\hat{\epsilon}_i$ are constructed in B^\dagger from the components in B of the unit vectors of B^* . For generality, B^\dagger was made a separate frame from B and B^* . If B^\dagger and B^* are allowed to coincide, it follows that the $\hat{\epsilon}_i^*$ are fixed, while the $\hat{\epsilon}_i$ vary in time. Further, the vectors $\hat{\epsilon}_i^*(t)$ can be simplified to

$$\hat{\epsilon}_1^*(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{\epsilon}_2^*(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{\epsilon}_3^*(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If ω is the angular velocity of B about B^* (the components are in B), then the $\hat{\epsilon}_i$ behave according to the law

$$\frac{d}{dt} \hat{\epsilon}_i = -\omega(t) \times \hat{\epsilon}_i \quad (1)$$

The minus sign is needed in equation (1) to describe the motion of the unit vectors of B^* as expressed in B . By straightforward differentiation it can be verified that equation (1) is satisfied if $\hat{e}_i(t)$ is defined as the formal infinite series

$$\begin{aligned}\hat{e}_i(t) = & \hat{e}_i(0) \\ & - \int_0^t \omega(t') \times \hat{e}_i(0) dt' \\ & + \int_0^t \omega(t') \times \int_0^{t'} \omega(t'') \times \hat{e}_i(0) dt'' dt' \\ & - \int_0^t \omega(t') \times \int_0^{t'} \omega(t'') \times \int_0^{t''} \omega(t''') \times \hat{e}_i(0) dt''' dt'' dt' \\ & + \dots\end{aligned}\tag{2}$$

where, because vector multiplication is not associative, products are performed from right to left. Equation (2) is thus a relation between basis vectors at time t .

Transformation of an arbitrary vector. — The vectors \mathbf{V} and \mathbf{V}^* are now written in terms of $\hat{e}_i(t)$ and $\hat{e}_i^*(t)$ as follows:

$$\begin{aligned}\mathbf{V}(t) &= \sum_{i=1}^3 v_i^*(t) \hat{e}_i(t) \\ \mathbf{V}^*(t) &= \sum_{i=1}^3 v_i(t) \hat{e}_i^*(t)\end{aligned}$$

Note that \mathbf{v} has components in B , relative to the unit vectors of B^* as expressed in B , equal to the components of \mathbf{V}^* in B^* . This is proper if \mathbf{V} and \mathbf{V}^* represent the same abstract quantity. Thus, the importance of this convention is that though \mathbf{V} and \mathbf{V}^* are constructed in B^* , information is obtained from equation (2) about components relative to the fundamental unit vectors of B .

Since the $\hat{e}_i(0)$ are related to the ϵ_i^* by the value of $A(t)$ at the time $t = 0$, the following is obtained for $v(t)$ in terms of $v^*(t)$:

$$\begin{aligned} v(t) = & A(0) \cdot v^*(t) - \int_0^t \omega(t') \times A(0) \cdot v^*(t) dt' \\ & + \int_0^t \omega(t') \times \int_0^{t'} \omega(t'') \times A(0) \cdot v^*(t) dt'' dt' - \dots \end{aligned} \quad (3)$$

Note that equation (3) is equivalent to a coordinate transformation. Thus, the usual attitude parameters have been avoided, and equation (3) may be substituted in the dynamical equations for attitude-dependent torques.

Dynamics

Functional dependence of gravity gradient torque. —At a given point in a gravitational field, a small body experiences a torque if the vector gradient is not zero. For a spherical Earth field, the torque is given by the expression

$$\ell = \frac{3G}{R} \hat{r} \times \hat{I} \cdot \hat{r}$$

In the equation above, \hat{r} is the unit vector from the Earth's center to the satellite and \hat{I} is the inertia dyadic of the satellite. For a circular orbit, the torque is thus a function of both attitude and time. If an oblate Earth is considered, the time variation in \hat{r} becomes more complex as the orbit precesses. No complications result from the field change caused by oblateness, however, because it is of a higher order. Thus, only the forced motion caused by a change in \hat{r} is of interest in this analysis.

Euler's equations. —When described on board, the rotational equation of motion for a space vehicle in the field of a spherical Earth is as follows:

$$\hat{I} \cdot \frac{d}{dt} \omega + \omega \times \hat{I} \cdot \omega = \frac{3G}{R} \hat{r} \times \hat{I} \cdot \hat{r} \quad (4)$$

Note that $\hat{r}(t)$ is not known, but $\hat{r}^*(t)$ (the direction in space) is.

The following parameter λ has a small value of the order of 10^{-7} for close circular orbits:

$$\lambda = \frac{3G}{R}$$

The torque expression can thus be written in terms of λ , as follows:

$$\lambda(\hat{\mathbf{r}} \times \mathbf{I} \cdot \hat{\mathbf{r}})$$

Complete Solution

Expansion in a small parameter. -The angular velocity, ω , is partitioned in terms of the parameter λ so that

$$\omega(t) = \omega_0(t) + \lambda \omega_1(t) + \lambda^2 \omega_2(t) + \dots \quad (5)$$

Similarly, $\hat{\mathbf{r}}(t)$ can be partitioned as a function of λ .

$$\hat{\mathbf{r}}(t) = \mathbf{r}_0(t) + \lambda \mathbf{r}_1(t) + \lambda^2 \mathbf{r}_2(t) + \dots \quad (6)$$

By using equation (3), the expression for each $\mathbf{r}_n(t)$ can be found as follows:

$$\begin{aligned} \mathbf{r}_0(t) = & \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) \\ & - \int_0^t \omega_0(t') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt' \\ & + \int_0^t \omega_0(t') \times \int_0^{t'} \omega_0(t'') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt'' dt' \\ & - \dots \end{aligned}$$

$$\begin{aligned}
\mathbf{r}_1(t) = & - \int_0^t \omega_1(t') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt' \\
& + \int_0^t \omega_1(t') \times \int_0^{t'} \omega_0(t'') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt'' dt' \\
& + \int_0^t \omega_0(t') \times \int_0^{t'} \omega_1(t'') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt'' dt' \\
& - \dots
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_2(t) = & - \int_0^t \omega_2(t') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt' \\
& + \int_0^t \omega_2(t') \times \int_0^{t'} \omega_0(t'') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt'' dt' \\
& + \int_0^t \omega_0(t') \times \int_0^{t'} \omega_2(t'') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt'' dt' \\
& + \int_0^t \omega_1(t') \times \int_0^{t'} \omega_1(t'') \times \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) dt'' dt' \\
& - \dots
\end{aligned}$$

and so on.

Equivalent system of linear differential equations. — The nonlinear equation (4) is reduced to a set of linear equations for the ω_n by equating in λ^n as follows:

$$\mathbf{I} : \frac{d}{dt} \omega_0 + \omega_0 \times \mathbf{I} \cdot \omega_0 = 0$$

$$\mathbf{I} \cdot \frac{d}{dt} \omega_1 + \omega_0 \times \mathbf{I} \cdot \omega_1 + \omega_1 \times \mathbf{I} \cdot \omega_0 = \mathbf{r}_0 \times \mathbf{I} \cdot \mathbf{r}_0$$

$$\mathbf{I} \cdot \frac{d}{dt} \omega_2 + \omega_0 \times \mathbf{I} \cdot \omega_2 + \omega_2 \times \mathbf{I} \cdot \omega_0 + \omega_1 \times \mathbf{I} \cdot \omega_1 = \mathbf{r}_0 \times \mathbf{I} \cdot \mathbf{r}_1 + \mathbf{r}_1 \times \mathbf{I} \cdot \mathbf{r}_0$$

.

.

.

$$\mathbf{I} \cdot \frac{d}{dt} \omega_n + \omega_0 \times \mathbf{I} \cdot \omega_n + \omega_n \times \mathbf{I} \cdot \omega_0 + \sum_{i=1}^{n-1} \omega_i \times \mathbf{I} \cdot \omega_{n-i} = \sum_{i=0}^{n-1} \mathbf{r}_i \times \mathbf{I} \cdot \mathbf{r}_{n-i-1}$$

with $n \rightarrow \infty$.

The first equation is nothing more than the free-body case first solved by Jacobi in terms of elliptic functions. The general equation for ω_n ($n = 1, 2, \dots$) can be put into the standard form

$$\frac{d}{dt}\omega_n = \Theta(t) \cdot \omega_n + \phi_n(t) \quad (7)$$

where

$$\begin{aligned} \Theta(t) &= I^{-1} \cdot (-\omega_0 \times I + E \times I \cdot \omega_0) \\ \phi_n(t) &= I^{-1} \cdot \left[\sum_{i=0}^{n-1} r_i \times I \cdot r_{n-i-1} - \sum_{i=1}^{n-1} \omega_i \times I \cdot \omega_{n-i} \right] \end{aligned}$$

with E as the idemfactor.

Formal integration in sequence. — As ω_0 is known, the equation for ω_1 can be solved, since r_0 depends only on ω_0 . It follows that ω_2 can be obtained, since r_1 depends only on ω_0 and ω_1 . Therefore, if at any point the solution to all ω_k ($0 \leq k < n$) has been obtained, equation (7) is left for ω_n , whose formal solution is verified by straightforward differentiation as

$$\omega_n(t) = \Gamma(t) \cdot \int_0^t \Gamma^{-1}(t') \cdot \phi_n(t') dt' \quad (8)$$

where

$$\begin{aligned} \Gamma(t) &= E + \int_0^t \Theta(t') dt' + \int_0^t \Theta(t') \cdot \int_0^{t'} \Theta(t'') dt'' dt' + \dots \\ \Gamma^{-1}(t) &= E - \int_0^t \Theta(t') dt' + \int_0^t \Theta(t') \cdot \int_0^{t'} \Theta(t'') dt'' dt' - \dots \end{aligned}$$

The initial conditions on the ω_n , $n > 0$, were chosen as zero in equation (8) because by equation (5) only the condition on ω_0 is needed.

Conclusion. — Equations (7) and (8) define a sequence of explicit operations; hence, $\omega(t)$ has been determined formally. Instantaneous attitude is determined with equation (2), and thus the complete problem is solved.

APPLICATION OF THEORY TO MORL

First Perturbation in Closed Form

Outline.— Assume an asymmetric vehicle in a precessing circular orbit with an initial spin, s , given to it at the principal axis of maximum moment of inertia. If $I_1 < I_2 < I_3$,

$$\omega_o(t) = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

because the vehicle would continue to spin about the larger axis if there were no disturbing torques.

The time variation of $\hat{\mathbf{r}}^*$ (the orbit unit radius as seen in the space frame), shown in figure 1, can be obtained by carrying the initial unit radius

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

at time $t = 0$, through the successive rotations - pt , i , and qt .

$$\mathbf{r}^*(t) = \begin{pmatrix} Cqt \, Cpt + Si \, Sqt \, Spt \\ -Cqt \, Spt + Si \, Sqt \, Cpt \\ Ci \, Sqt \end{pmatrix} \quad (9)$$

where $C \equiv \cos$, $S \equiv \sin$.

From the orientation of the vehicle for $t = 0$ in figure 1, it follows that $\mathbf{A}(0)$ has the form

$$\mathbf{A}(0) = \begin{pmatrix} -S\phi & C\phi & 0 \\ -C\theta C\phi & -C\theta S\phi & S\theta \\ S\theta C\phi & S\theta S\phi & C\theta \end{pmatrix} \quad (10)$$

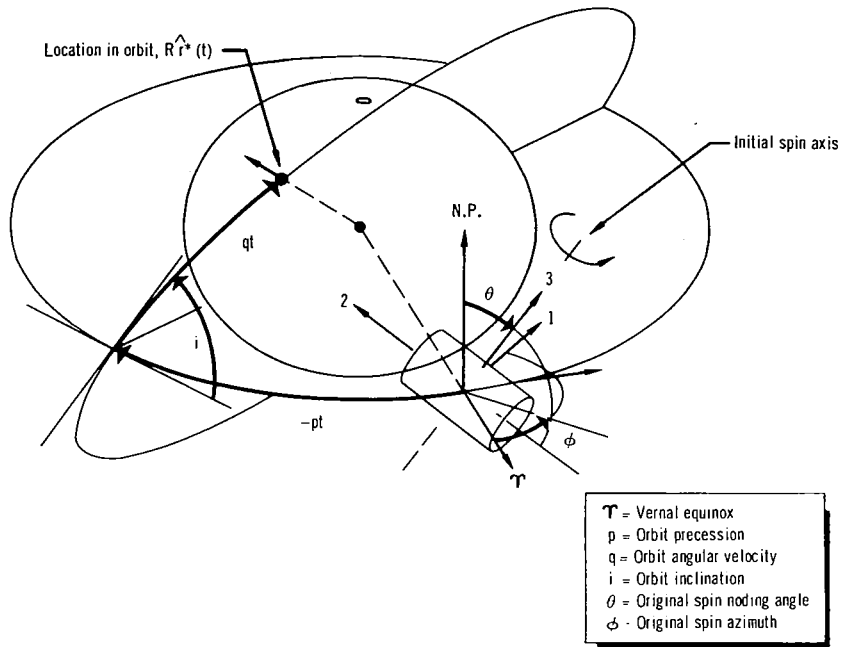


Figure 1. Orbital Motion and Vehicle Initial Attitude

Because of the construction of $\mathbf{r}_0(t)$ and the particular choice of $\omega_0(t)$, the following definition can be made:

$$\mathbf{a}(t) = \mathbf{A}(0) \cdot \hat{\mathbf{r}}^*(t) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

The first term in $\mathbf{r}_0(t)$ will be \mathbf{a} , while the second, third, etc., terms are, in order, the following:

$$-\int_0^t \omega_0 \times \mathbf{a} dt' = -st \begin{pmatrix} -a_2 \\ a_1 \\ 0 \end{pmatrix} = -st\beta$$

$$\int_0^t \omega_0 \times \int_0^{t'} \omega_0 \times \mathbf{a} dt'' dt' = \int_0^t \omega_0 \times (st'\beta) dt'$$

$$= \frac{1}{2} s^2 t^2 \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} = -\frac{1}{2} s^2 t^2 \gamma$$

$$\begin{aligned}
-\int_0^t \omega_0 \times \int_0^{t'} \omega_0 \times \int_0^{t''} \omega_0 \times a \, dt''' \, dt'' \, dt' &= -\int_0^t \omega_0 \times \left(-\frac{1}{2} s^2 t'^2 \gamma \right) dt' \\
&= \frac{1}{2 \cdot 3} s^3 t^3 \begin{pmatrix} -a_2 \\ a_1 \\ 0 \end{pmatrix} \\
&= \frac{1}{2 \cdot 3} s^3 t^3 \beta
\end{aligned}$$

and so on.

The whole expression for $r_0(t)$ is divided into odd powers of t multiplied by the vector β , and even powers of t (greater than zero) multiplied by the vector γ , with alternations in sign for each, as follows:

$$\begin{aligned}
r_0(t) &= a(t) - \sum_{k=0}^{\infty} (-1)^k \frac{s^{2k+1} t^{2k+1}}{(2k+1)!} \beta(t) \\
&\quad + \sum_{k=1}^{\infty} (-1)^k \frac{s^{2k} t^{2k}}{(2k)!} \gamma(t)
\end{aligned}$$

$$r_0(t) = a(t) - \beta(t) \sin st + \gamma(t) [\cos st - 1] \quad (11)$$

The value of $\phi_1(t)$ (see equation (7)) can be written preparatory to solving for $\omega_1(t)$, since the components of a , β , and γ (that is, the a_i) are known.

$$\phi_1(t) = I^{-1} \cdot [r_0(t) \times I \cdot r_0(t)]$$

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$$I^{-1} = \begin{pmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{pmatrix}$$

hence

$$\phi_1(t) = \begin{pmatrix} \frac{I_3 - I_2}{I_1} & r_{02} & r_{03} \\ \frac{I_1 - I_3}{I_2} & r_{03} & r_{01} \\ \frac{I_2 - I_1}{I_3} & r_{01} & r_{02} \end{pmatrix} \quad (12)$$

where r_{0i} is the i^{th} component of \mathbf{r}_0 .

The value of $\boldsymbol{\theta}(t)$ from equation (7) is obtained next to construct Γ and Γ^{-1} (see equation(8)) and to solve finally for ω_1 in equation (8).

$$\boldsymbol{\theta}(t) = \begin{pmatrix} 0 & \left(\frac{I_2 - I_3}{I_1}\right)s & 0 \\ \left(\frac{I_3 - I_1}{I_2}\right)s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Some of the first few integrals in the definitions of Γ and Γ^{-1} from equation (8) are formed as follows:

$$\int_0^t \boldsymbol{\theta} dt' = t \begin{pmatrix} 0 & \mu & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = t\boldsymbol{\theta}$$

$$\int_0^t \boldsymbol{\theta} \cdot \int_0^{t'} \boldsymbol{\theta} dt'' dt' = \int_0^t \boldsymbol{\theta} \cdot (t' \boldsymbol{\theta}) dt' = \boldsymbol{\theta}^2 \int_0^t t' dt'$$

$$= \frac{t^2}{2} \rho \mu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \rho \mu \frac{t^2}{2} \mathbf{H}$$

$$\int_0^t \boldsymbol{\theta} \cdot \int_0^{t'} \boldsymbol{\theta} \cdot \int_0^{t''} \boldsymbol{\theta} dt''' dt'' dt' = \int_0^t \boldsymbol{\theta} \cdot (\rho \mu \frac{t'^2}{2} \mathbf{H}) dt'$$

$$= \rho \mu \frac{t^3}{2 \cdot 3} \boldsymbol{\theta}$$

and so on.

Thus, Γ and Γ^{-1} are expressions which divide themselves into odd powers of t times Θ and even powers of t (greater than zero) times H .

$$\begin{aligned}\Gamma(t) &= E + \frac{1}{\sqrt{\rho\mu}} \sum_{k=0}^{\infty} \frac{(\sqrt{\rho\mu} t)^{2k+1}}{(2k+1)!} \Theta + \sum_{k=1}^{\infty} \frac{(\sqrt{\rho\mu} t)^{2k}}{(2k)!} H \\ &= E + H \left[\cosh \sqrt{\rho\mu} t - 1 \right] + \frac{1}{\sqrt{\rho\mu}} \Theta \sinh \sqrt{\rho\mu} t \\ \Gamma^{-1}(t) &= E + H \left[\cosh \sqrt{\rho\mu} t - 1 \right] - \frac{1}{\sqrt{\rho\mu}} \Theta \sinh \sqrt{\rho\mu} t\end{aligned}$$

If $\rho\mu < 0$, then $(I_3 - I_1)(I_3 - I_2) > 0$ and the hyperbolic functions become circular, thus stating that $\omega_1(t)$ is bounded. When I_3 is the largest moment of inertia, as is assumed in this study, the motion of the free body is known to be stable. Hence, $\rho\mu < 0$, and, by a well known identity,

$$\Gamma(t) = E + H \left[\cos \sqrt{-\rho\mu} t - 1 \right] + \frac{1}{\sqrt{-\rho\mu}} \Theta \sin \sqrt{-\rho\mu} t \quad (13)$$

$$\Gamma^{-1}(t) = E + H \left[\cos \sqrt{-\rho\mu} t - 1 \right] - \frac{1}{\sqrt{-\rho\mu}} \Theta \sin \sqrt{-\rho\mu} t \quad (14)$$

The problem now can be solved for $\omega_1(t)$ in equation (8). The solution will result in an enormous buildup of sums of products of circular functions because of the structure of equations (9) and (10), which form equation (11), and the structure of equation (12), which is formed from equation (11), premultiplication by equation (14), integration, and premultiplication by equation (13). To handle this complex situation efficiently, a computer algebra for circular functions was invented; this is explained in the next section.

Algebra of circular functions.— It has been stated that the solution of $\omega_1(t)$ involves the manipulation of a large number of circular terms resulting from products of sums. Because circular functions can be either cosines or sines, some system of identification is needed to keep track of each term. The situation is more involved when the sum of products is considered, for example, $\sin x_1 t \cos x_2 t$ added to $\cos x_3 t \sin x_4 t \sin x_5 t$. Uniform notation was used in which only cosines of t are considered. Thus, terms can be identified with subscripts without regard to the kind of function. If products of cosines are generated, they can be reduced to sums of cosines by a redefinition of constants. Thus,

$$\cos x_1 t \cos x_2 t = \frac{1}{2} \left[\cos (x_1 + x_2) t + \cos (x_1 - x_2) t \right] \quad (15)$$

Cosines were chosen, since, of the two functions, only they reproduce themselves in the above type of decomposition. The elimination of sines is not straightforward, and a superscript was employed with an ensuing set of algebraic rules of combination. A sine function is converted to a cosine as follows:

$$\begin{aligned}\sin x^0 t &= \cos (x^0 t - \frac{\pi}{2}) \\ &= \cos x^1 t\end{aligned}$$

Clearly, x^1 is a constant only in symbolism and will be handled as such. The superscript 1 indicates that a subtraction of $\pi/2$ is hidden. A superscript of -2, for example, would mean subtraction of $-\pi$ or, equally, addition of π . It should also be noted that x^k behaves numerically as if it were equal to x^0 under operations of integration or differentiation, except that for integration the value of k must be known to evaluate the limits. If sums such as those in $\cos (x_1^i \pm x_2^j)t$ are to be performed, the superscripts (or indexes) obey the following rule:

$$x_1^i \pm x_2^j = (x_1 \pm x_2)^{i \pm j} \quad (16)$$

In carrying out the explicit determination of $\omega_1(t)$, the constants in $a_i \cos x_i t$ are redefined half a dozen times, and each time i runs from 1 to a larger number. At each step, it is easy to program the defining relations on a computer to obtain the values of a_i , x_i , and the index. If all details were of interest, it would be necessary to do the algebra on a computer because each component of $\omega_1(t)$ is made up of 1,298 terms. However, only a small number of terms (those associated with the orbit precession) are significantly large for spin-axis wander, thus the rest can be eliminated.

Angular velocity. -- The problem here is to obtain a more detailed description of $\omega_1(t)$ in closed form from the preceding outline by use of the algebra of circular functions just developed. The following operations are to be performed in order: $A(0) \cdot r^*(t) = a(t)$ is obtained from equations (9) and (10), then $r_0(t)$ is obtained from equation (11), $\phi_1(t)$ is obtained from equation (12), and $\omega_1(t)$ is obtained from equation (8) using Γ and Γ^{-1} in equations (13) and (14). The results up to $\phi_1(t)$ are given below (see equation (20)) for $\omega_1(t)$ to show the large number of circular functions required.

$$\mathbf{a}(t) = \begin{pmatrix} \sum_{j=1}^9 B_{1,j} \cos x_j t \\ \sum_{j=1}^9 B_{2,j} \cos x_j t \\ \sum_{j=1}^9 B_{3,j} \cos x_j t \end{pmatrix} \quad (17)$$

$$\mathbf{r}_o(t) = \begin{pmatrix} \sum_{j=1}^{36} D_{1,j} \cos a_j t \\ \sum_{j=1}^{36} D_{2,j} \cos a_j t \\ \sum_{j=1}^9 B_{3,j} \cos x_j t \end{pmatrix} \quad (18)$$

$$\phi_1(t) = \begin{pmatrix} \sum_{j=1}^{648} E_{1,j} \cos \epsilon_j t \\ \sum_{j=1}^{648} E_{2,j} \cos \epsilon_j t \\ \sum_{j=1}^{2592} F_j \cos f_j t \end{pmatrix} \quad (19)$$

The defining relations from equations (9) and (10) for $\alpha(t)$ are, with the use of equation (15),

$$\cos y_1 t = \cos pt$$

$$\cos y_2 t = \cos qt$$

$$\cos y_3 t = \sin pt$$

$$\cos y_4 t = \sin qt$$

$$x_1 = y_1 + y_2$$

$$x_4 = y_3 - y_4$$

$$x_7 = y_1 + y_4$$

$$x_2 = y_1 - y_2$$

$$x_5 = y_3 + y_2$$

$$x_8 = y_1 - y_4$$

$$x_3 = y_3 + y_4$$

$$x_6 = y_3 - y_2$$

$$x_9 = y_4$$

$$B_{1,1} = \frac{1}{2} A_{11}$$

$$B_{2,1} = \frac{1}{2} A_{21}$$

$$B_{3,1} = \frac{1}{2} A_{31}$$

$$B_{1,2} = \frac{1}{2} A_{11}$$

$$B_{2,2} = \frac{1}{2} A_{21}$$

$$B_{3,2} = \frac{1}{2} A_{31}$$

$$B_{1,3} = \frac{1}{2} C_1 A_{11}$$

$$B_{2,3} = \frac{1}{2} C_1 A_{21}$$

$$B_{3,3} = \frac{1}{2} C_1 A_{31}$$

$$B_{1,4} = \frac{1}{2} C_1 A_{11}$$

$$B_{2,4} = \frac{1}{2} C_1 A_{21}$$

$$B_{3,4} = \frac{1}{2} C_1 A_{31}$$

$$B_{1,5} = -\frac{1}{2} A_{12}$$

$$B_{2,5} = -\frac{1}{2} A_{22}$$

$$B_{3,5} = -\frac{1}{2} A_{32}$$

$$B_{1,6} = -\frac{1}{2} A_{12}$$

$$B_{2,6} = -\frac{1}{2} A_{22}$$

$$B_{3,6} = -\frac{1}{2} A_{32}$$

$$B_{1,7} = \frac{1}{2} C_1 A_{12}$$

$$B_{2,7} = \frac{1}{2} C_1 A_{22}$$

$$B_{3,7} = \frac{1}{2} C_1 A_{32}$$

$$B_{1,8} = \frac{1}{2} C_1 A_{12}$$

$$B_{2,8} = \frac{1}{2} C_1 A_{22}$$

$$B_{3,8} = \frac{1}{2} C_1 A_{32}$$

$$B_{1,9} = C_2 A_{13}$$

$$B_{2,9} = C_2 A_{23}$$

$$B_{3,9} = C_2 A_{33}$$

with

$$C_1 = \sin i$$

$$C_2 = \cos i$$

and

$$A_{11} = -\sin \phi$$

$$A_{12} = \cos \phi$$

$$A_{13} = 0$$

$$A_{21} = -\cos \theta \cos \phi$$

$$A_{22} = -\cos \theta \sin \phi$$

$$A_{23} = \sin \theta$$

$$A_{31} = \sin \theta \cos \phi$$

$$A_{32} = \sin \theta \sin \phi$$

$$A_{33} = \cos \theta$$

so that the result is (17).

The defining relations for $\mathbf{r}_0(t)$ (see equation (11)) are

$$\cos z_1 t = \cos st$$

$$\cos z_2 t = \sin st$$

$$a_i = x_i + z_1$$

$$a_{9+i} = x_i - z_1$$

$$i = 1, 2, \dots, 9$$

$$a_{18+i} = x_i + z_2$$

$$a_{27+i} = x_i - z_2$$

$$D_{1,i'} = \frac{1}{2} B_{1,i}$$

$$D_{2,i'} = \frac{1}{2} B_{2,i}$$

$$i = 1, 2, \dots, 9$$

$$i' = i + 9j$$

$$D_{1,18+i'} = \frac{1}{2} B_{2,i}$$

$$D_{2,18+i'} = -\frac{1}{2} B_{1,i}$$

$$j = 0, 1$$

resulting in equation (18).

The analogous relations for ϕ_1 (see equation (12)) are

$$\begin{aligned}
\epsilon_i &= x_1 + a_i & \epsilon_{36+i} &= x_1 - a_i & i &= 1, 2, \dots, 36 \\
\epsilon_{72+i} &= x_2 + a_i & \epsilon_{108+i} &= x_2 - a_i \\
\epsilon_{144+i} &= x_3 + a_i & \epsilon_{180+i} &= x_3 - a_i \\
\epsilon_{216+i} &= x_4 + a_i & \epsilon_{252+i} &= x_4 - a_i \\
\epsilon_{288+i} &= x_5 + a_i & \epsilon_{324+i} &= x_5 - a_i \\
\epsilon_{360+i} &= x_6 + a_i & \epsilon_{396+i} &= x_6 - a_i \\
\epsilon_{432+i} &= x_7 + a_i & \epsilon_{468+i} &= x_7 - a_i \\
\epsilon_{504+i} &= x_8 + a_i & \epsilon_{540+i} &= x_8 - a_i \\
\epsilon_{576+i} &= x_9 + a_i & \epsilon_{612+i} &= x_9 - a_i
\end{aligned}$$

$$\begin{aligned}
E_{1,i'} &= \frac{1}{2} G_1 B_{3,1} D_{2,i} & E_{2,i'} &= \frac{1}{2} G_2 B_{3,1} D_{1,i} & i &= 1, \dots, 36 \\
E_{1,72+i'} &= \frac{1}{2} G_1 B_{3,2} D_{2,i} & E_{2,72+i'} &= \frac{1}{2} G_2 B_{3,2} D_{1,i} & i' &= i + 36j \\
E_{1,144+i'} &= \frac{1}{2} G_1 B_{3,3} D_{2,i} & E_{2,144+i'} &= \frac{1}{2} G_2 B_{3,3} D_{1,i} & j &= 0, 1 \\
E_{1,216+i'} &= \frac{1}{2} G_1 B_{3,4} D_{2,i} & E_{2,216+i'} &= \frac{1}{2} G_2 B_{3,4} D_{1,i} \\
E_{1,288+i'} &= \frac{1}{2} G_1 B_{3,5} D_{2,i} & E_{2,288+i'} &= \frac{1}{2} G_2 B_{3,5} D_{1,i} \\
E_{1,360+i'} &= \frac{1}{2} G_1 B_{3,6} D_{2,i} & E_{2,360+i'} &= \frac{1}{2} G_2 B_{3,6} D_{1,i} \\
E_{1,432+i'} &= \frac{1}{2} G_1 B_{3,7} D_{2,i} & E_{2,432+i'} &= \frac{1}{2} G_2 B_{3,7} D_{1,i} \\
E_{1,504+i'} &= \frac{1}{2} G_1 B_{3,8} D_{2,i} & E_{2,504+i'} &= \frac{1}{2} G_2 B_{3,8} D_{1,i} \\
E_{1,576+i'} &= \frac{1}{2} G_1 B_{3,9} D_{2,i} & E_{2,576+i'} &= \frac{1}{2} G_2 B_{3,9} D_{1,i}
\end{aligned}$$

where

$$G_1 = \frac{I_3 - I_2}{I_1}$$

$$G_2 = \frac{I_1 - I_3}{I_2}$$

thus giving the first two components of equation (19).

No calculations will be made of F_j or f_j in equation (19) because it can be seen from the following formulations that they are unnecessary. The ϵ_j and $E_{i,j}$ can be written in a more compact form as

$$\epsilon_{36[2(j-1)]+i} = x_j + a_i \quad i = 1, 2, \dots, 36$$

$$\epsilon_{36[2(j-1)+1]+i} = x_j - a_i \quad j = 1, 2, \dots, 9$$

$$E_{1, 36[j+2(k-1)]+i} = \frac{1}{2} G_1 B_{3,k} D_{2,i} \quad i = 1, \dots, 36$$

$$j = 0, 1$$

$$E_{2, 36[j+2(k-1)]+i} = \frac{1}{2} G_2 B_{3,k} D_{1,i} \quad k = 1, \dots, 9$$

The index associated with ϵ_k has been neglected thus far; however, because of equation (16), the calculation of the index is seen to be straightforward. In fact, the recursive relations for the indexes have the same form as the defining relations for the x_i , a_i , and the ϵ_i , with the following initial conditions:

$$\text{index}(y_1) = 0$$

$$\text{index}(y_2) = 0$$

$$\text{index}(y_3) = 1$$

$$\text{index}(y_4) = 1$$

$$\text{index}(z_1) = 0$$

$$\text{index}(z_2) = 1$$

Next, ϕ_1 in equation (12) is used with the values of Γ and Γ^{-1} in equations (13) and (14) to prepare to solve for a_1 in equation (8).

$$\Gamma^{-1}(t) \cdot \phi_1(t) = \begin{pmatrix} \phi_{11} \cos \sqrt{-\rho\mu} t + \sqrt{-\frac{\mu}{\rho}} \phi_{12} \sin \sqrt{-\rho\mu} t \\ \phi_{12} \cos \sqrt{-\rho\mu} t + \sqrt{-\frac{\rho}{\mu}} \phi_{11} \sin \sqrt{-\rho\mu} t \\ \phi_{13} \end{pmatrix}$$

To obtain $\omega_1(t)$, the above equation must be integrated, which, for typical terms (the index of each ϵ_i must be observed), results in the following:

$$\begin{aligned} \int_0^t \cos \epsilon t' \cos \sqrt{-\rho\mu} t' dt' &= \frac{\sin(\epsilon + \sqrt{-\rho\mu}) t'}{2(\epsilon + \sqrt{-\rho\mu})} + \frac{\sin(\epsilon - \sqrt{-\rho\mu}) t'}{2(\epsilon - \sqrt{-\rho\mu})} \Big|_0^t \\ &= \frac{\sin(\epsilon + \sqrt{-\rho\mu}) t - \sin\{(\epsilon + \sqrt{-\rho\mu}) 0\}}{2(\epsilon + \sqrt{-\rho\mu})} \\ &\quad + \frac{\sin(\epsilon - \sqrt{-\rho\mu}) t - \sin\{(\epsilon - \sqrt{-\rho\mu}) 0\}}{2(\epsilon - \sqrt{-\rho\mu})} \\ \int_0^t \cos \epsilon t' \sin \sqrt{-\rho\mu} t' dt' &= -\frac{\cos(\epsilon + \sqrt{-\rho\mu}) t'}{2(\epsilon + \sqrt{-\rho\mu})} + \frac{\cos(\epsilon - \sqrt{-\rho\mu}) t'}{2(\epsilon - \sqrt{-\rho\mu})} \Big|_0^t \\ &= -\frac{\cos(\epsilon + \sqrt{-\rho\mu}) t - \cos\{(\epsilon + \sqrt{-\rho\mu}) 0\}}{2(\epsilon + \sqrt{-\rho\mu})} \\ &\quad + \frac{\cos(\epsilon - \sqrt{-\rho\mu}) t - \cos\{(\epsilon - \sqrt{-\rho\mu}) 0\}}{2(\epsilon - \sqrt{-\rho\mu})} \end{aligned}$$

The circular functions are expanded and combined into the more usable forms

$$\begin{aligned}
 \frac{\sin(\epsilon + \sqrt{-\rho\mu})t}{2(\epsilon + \sqrt{-\rho\mu})} + \frac{\sin(\epsilon - \sqrt{-\rho\mu})t}{2(\epsilon - \sqrt{-\rho\mu})} &= \frac{\sin \epsilon t \cos \sqrt{-\rho\mu}t + \cos \epsilon t \sin \sqrt{-\rho\mu}t}{2(\epsilon + \sqrt{-\rho\mu})} \\
 &+ \frac{\sin \epsilon t \cos \sqrt{-\rho\mu}t - \cos \epsilon t \sin \sqrt{-\rho\mu}t}{2(\epsilon - \sqrt{-\rho\mu})} \\
 &= \frac{\epsilon}{\epsilon^2 + \rho\mu} \sin \epsilon t \cos \sqrt{-\rho\mu}t \\
 &- \frac{\sqrt{-\rho\mu}}{\epsilon^2 + \rho\mu} \cos \epsilon t \sin \sqrt{-\rho\mu}t
 \end{aligned}$$

and

$$\begin{aligned}
 -\frac{\cos(\epsilon + \sqrt{-\rho\mu})t}{2(\epsilon + \sqrt{-\rho\mu})} + \frac{\cos(\epsilon - \sqrt{-\rho\mu})t}{2(\epsilon - \sqrt{-\rho\mu})} &= -\frac{\cos \epsilon t \cos \sqrt{-\rho\mu}t - \sin \epsilon t \sin \sqrt{-\rho\mu}t}{2(\epsilon + \sqrt{-\rho\mu})} \\
 &+ \frac{\cos \epsilon t \cos \sqrt{-\rho\mu}t + \sin \epsilon t \sin \sqrt{-\rho\mu}t}{2(\epsilon - \sqrt{-\rho\mu})} \\
 &= -\frac{\sqrt{-\rho\mu}}{\epsilon^2 + \rho\mu} \cos \epsilon t \cos \sqrt{-\rho\mu}t \\
 &+ \frac{\epsilon}{\epsilon^2 + \rho\mu} \sin \epsilon t \sin \sqrt{-\rho\mu}t
 \end{aligned}$$

The same operations are performed for the terms evaluated at $t = 0$. Now the complete integrals can be written as follows:

$$\int_0^t \phi_{11}(t') \cos \sqrt{-\rho\mu}t' dt' = A_1(t) \cos \sqrt{-\rho\mu}t + B_1(t) \sin \sqrt{-\rho\mu}t - A_1(0)$$

where

$$A_1(t) = \sum_{j=1}^{648} \frac{\epsilon_j}{\epsilon_j^2 + \rho\mu} E_{1,j} \sin \epsilon_j t$$

$$B_1(t) = - \sum_{j=1}^{648} \frac{\sqrt{-\rho\mu}}{\epsilon_j^2 + \rho\mu} E_{1,j} \cos \epsilon_j t$$

and

$$\int_0^t \phi_{12}(t') \sin \sqrt{-\rho\mu} t' dt' = B_2(t) \cos \sqrt{-\rho\mu} t + A_2(t) \sin \sqrt{-\rho\mu} t - B_2(o)$$

with

$$B_2(t) = - \sum_{j=1}^{648} \frac{\sqrt{-\rho\mu}}{\epsilon_j^2 + \rho\mu} E_{2,j} \cos \epsilon_j t$$

$$A_2(t) = \sum_{j=1}^{648} \frac{\epsilon_j}{\epsilon_j^2 + \rho\mu} E_{2,j} \sin \epsilon_j t$$

Therefore,

$$\int_0^t \Gamma^{-1}(t') \cdot \phi_1(t') dt'$$

may be written as

$$\begin{pmatrix} \left[A_1(t) + \sqrt{-\frac{\mu}{\rho}} B_2(t) \right] \cos \sqrt{-\rho\mu} t + \left[B_1(t) + \sqrt{-\frac{\mu}{\rho}} A_2(t) \right] \sin \sqrt{-\rho\mu} t - \left[A_1(o) + \sqrt{-\frac{\mu}{\rho}} B_2(o) \right] \\ \left[A_2(t) + \sqrt{-\frac{\rho}{\mu}} B_1(t) \right] \cos \sqrt{-\rho\mu} t + \left[B_2(t) + \sqrt{-\frac{\rho}{\mu}} A_1(t) \right] \sin \sqrt{-\rho\mu} t - \left[A_2(o) + \sqrt{-\frac{\rho}{\mu}} B_1(o) \right] \\ \int_0^t \phi_{13}(t') dt' \end{pmatrix}$$

$$= \begin{pmatrix} \psi_1(t) \cos \sqrt{-\rho\mu} t + \sqrt{-\frac{\mu}{\rho}} \psi_2(t) \sin \sqrt{-\rho\mu} t \\ \psi_2(t) \cos \sqrt{-\rho\mu} t + \sqrt{-\frac{\rho}{\mu}} \psi_1(t) \sin \sqrt{-\rho\mu} t \\ \psi_3(t) \end{pmatrix} + \begin{pmatrix} \psi_1(o) \\ \psi_2(o) \\ 0 \end{pmatrix}$$

for

$$\psi_1(t) = A_1(t) + \sqrt{-\frac{\mu}{\rho}} B_2(t)$$

$$\psi_2(t) = A_2(t) + \sqrt{-\frac{\rho}{\mu}} B_1(t)$$

$$\psi_3(t) = \int_0^t \phi_{13}(t') dt'$$

It then follows that

$$\int_0^t \Gamma^{-1}(t') \cdot \phi_1(t') dt' = \Gamma^{-1}(t) \cdot \psi(t) + \psi(o)$$

is true,

Thus, upon multiplying on the left by $\Gamma(t)$, a simple relation for $\omega_1(t)$

$$\omega_1(t) = \psi(t) + \Gamma(t) \cdot \psi(o)$$

is obtained and is written in terms of the cosine notation.

$$\omega_1(t) = \begin{pmatrix} \sum_{j=1}^{1298} G_{1,j} \cos \gamma_j t \\ \sum_{j=1}^{1298} G_{2,j} \cos \gamma_j t \\ \int_0^t \phi_{13}(t') dt' \end{pmatrix} \quad (20)$$

for

$$\begin{aligned} \cos \gamma_j t &= \cos \epsilon_j t & j &= 1, 2, \dots, 648 \\ \cos (\gamma_{648+j}) t &= \sin \epsilon_j t \\ \cos (\gamma_{1297}) t &= \cos \sqrt{-\rho\mu} t \\ \cos (\gamma_{1298}) t &= \sin \sqrt{-\rho\mu} t \end{aligned}$$

$$G_{1,j} = -\frac{\mu}{\epsilon_j^2 + \rho\mu} E_{2,j} \quad j = 1, 2, \dots, 648$$

$$G_{1,648+j} = \frac{\epsilon_j}{\epsilon_j^2 + \rho\mu} E_{1,j}$$

$$G_{1,1297} = \sum_{i=1}^{648} \frac{1}{\epsilon_j^2 + \rho\mu} \left[-\epsilon_i E_{1,i} \sin(\epsilon_i o) + \mu E_{2,i} \cos(\epsilon_i o) \right]$$

$$G_{1,1298} = \sum_{i=1}^{648} \frac{1}{\epsilon_j^2 + \rho\mu} \left[-\sqrt{-\frac{\mu}{\rho}} \epsilon_i E_{2,i} \sin(\epsilon_i o) + \sqrt{-\rho\mu} E_{1,i} \cos(\epsilon_i o) \right]$$

$$G_{2,j} = -\frac{\rho}{\epsilon_j^2 + \rho\mu} E_{1,j} \quad j = 1, 2, \dots, 648$$

$$G_{2,648+j} = \frac{\epsilon_j}{\epsilon_j^2 + \rho\mu} E_{2,j}$$

$$G_{2,1297} = \sum_{i=1}^{648} \frac{1}{\epsilon_j^2 + \rho\mu} \left[-\epsilon_i E_{2,i} \sin(\epsilon_i o) + \rho E_{1,i} \cos(\epsilon_i o) \right]$$

$$G_{2,1298} = \sum_{i=1}^{648} \frac{1}{\epsilon_j^2 + \rho\mu} \left[-\sqrt{-\frac{\rho}{\mu}} \epsilon_i E_{1,i} \sin(\epsilon_i o) + \sqrt{-\rho\mu} E_{2,i} \cos(\epsilon_i o) \right]$$

Only the first two components of $\omega_1(t)$ are required, as will be shown; each has now been expressed by recursive relations as a sum of 1,298 circular functions. The amount of work saved by the algebra of circular functions is apparent, for to have determined $\omega_1(t)$ by brute-force manipulations would have required many pages and the possibility of error would have been greater. Note that $|\lambda \omega_1(t)| \ll |\omega_o(t)|$ because of the formation of the coefficient leading to the $G_{1,j}$ and the smallness of λ . This shows that the coning about the instantaneous angular momentum is very small, although the angular momentum has a larger but slow wander in space (much like a fast top). The angular-momentum wander as seen on board the vehicle must be determined next.

Spin-axis direction. — In the last section, a closed form expression for the first perturbation of angular velocity was determined as $\omega(t) = \omega_o(t) + \lambda \omega_1(t)$. Now the spin-axis wander caused by the additional term $\lambda \omega_1(t)$ must be determined. This is best accomplished by expressing the initial direction in space, $\hat{h}(o)$, as seen on board the vehicle at time t .

$$\hat{h}(o) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Substituting the above in equation (3),

$$\hat{h}(t) = \hat{h}(o) - \int_0^t \omega(t') \times \hat{h}(o) dt' + \int_0^t \omega(t') \times \int_0^{t'} \omega(t'') \times \hat{h}(o) dt'' dt' - \dots$$

where

$$\omega(t) = s \begin{pmatrix} a_1(t) \\ a_2(t) \\ 1 + a_3(t) \end{pmatrix}$$

with

$$a_i(t) = \frac{\lambda}{s} \sum_{j=1}^{12.98} G_{i,j} \cos \gamma_j t \quad i = 1, 2$$

$$a_3(t) = \frac{\lambda}{s} \int_0^t \phi_{13}(t') dt'$$

Because $a_i < 1$, products can be neglected in evaluating each term in the expression of $\hat{h}(t)$.

$$\hat{h}(t) = \sum_{k=0}^{\infty} \delta_k = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

In the definitions of δ_k below, the number of ones in front of the α_i represents the number of integrations to be performed, for example,

$$-1111\alpha_1 \equiv - \int_0^t \int_0^{t'} \int_0^{t''} \int_0^{t'''} \alpha_1(t^{iv}) dt^{iv} dt''' dt'' dt'$$

so that

$$\delta_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\delta_1 = -s \begin{pmatrix} 1\alpha_2 \\ -1\alpha_1 \\ 0 \end{pmatrix}$$

$$\delta_2 = s^2 \begin{pmatrix} 11\alpha_1 \\ 11\alpha_2 \\ 0 \end{pmatrix}$$

$$\delta_3 = -s^3 \begin{pmatrix} -111\alpha_2 \\ 111\alpha_1 \\ 0 \end{pmatrix}$$

$$\delta_4 = s^4 \begin{pmatrix} -1111\alpha_1 \\ -1111\alpha_2 \\ 0 \end{pmatrix}$$

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Note that as the multiple integrations progress, more powers of γ_j are thrown into the denominator. As the integrals vary from 0 to t , the index associated with γ_j must be observed for evaluating the limit at $t = 0$.

$$\int_0^t \cos \gamma t' dt' = \frac{1}{\gamma} [\sin \gamma t - \sin (\gamma_0)]$$

$$\int_0^t \int_0^{t'} \cos \gamma t'' dt'' dt' = -\frac{1}{\gamma^2} [\cos \gamma t - \cos (\gamma_0)] - \frac{t}{\gamma} \sin (\gamma_0)$$

$$\int_0^t \int_0^{t'} \int_0^{t''} \cos \gamma t''' dt''' dt'' dt' = -\frac{1}{\gamma^3} [\sin \gamma t - \sin (\gamma_0)] + \frac{t}{\gamma^2} \cos (\gamma_0) - \frac{t^2}{2\gamma} \sin (\gamma_0)$$

When the required integrations are performed, the algebra can be arranged as follows:

$$\begin{aligned} h_I(t) = \sum_{j=1}^{1298} \frac{\lambda}{s} G_{2,j} \left\{ \sin \gamma_j t \left(\frac{s}{\gamma_j} + \frac{s^3}{\gamma_j^3} + \frac{s^5}{\gamma_j^5} + \dots \right) \right. \\ + \sin (\gamma_j 0) \left[\frac{s}{\gamma_j} \left(1 - \frac{s^2 t^2}{2} + \frac{s^4 t^4}{4!} - \dots \right) \right. \\ + \frac{s^3}{\gamma_j^3} \left(1 - \frac{s^2 t^2}{2} + \frac{s^4 t^4}{4!} - \dots \right) \\ + \frac{s^5}{\gamma_j^5} \left(1 - \frac{s^2 t^2}{2} + \frac{s^4 t^4}{4!} - \dots \right) + \dots \Big] \\ + \cos (\gamma_j 0) \left[\frac{s^2}{\gamma_j^2} \left(st - \frac{s^3 t^3}{3!} + \dots \right) \right. \\ + \frac{s^4}{\gamma_j^4} \left(st - \frac{s^3 t^3}{3!} + \dots \right) + \dots \Big] \Big\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{1298} \frac{\lambda}{s} G_{1,j} \left\{ \cos \gamma_j t \left(\frac{s^2}{\gamma_j^2} + \frac{s^4}{\gamma_j^4} + \frac{s^6}{\gamma_j^6} + \dots \right) \right. \\
& \quad + \sin (\gamma_j o) \left[\frac{s}{\gamma_j} \left(-st + \frac{s^3 t^3}{3!} - \dots \right) \right. \\
& \quad \quad \left. \left. + \frac{s^3}{\gamma_j^3} \left(-st + \frac{s^3 t^3}{3!} - \dots \right) + \dots \right] \right. \\
& \quad \left. + \cos (\gamma_j o) \left[\frac{s^2}{\gamma_j^2} \left(1 - \frac{s^2 t^2}{2} + \dots \right) + \dots \right] \right\}
\end{aligned}$$

A similar expression results for $h_2(t)$, so that when the infinite series are summed up, the following expressions are obtained for $h_1(t)$ and $h_2(t)$:

$$\begin{aligned}
h_1(t) = & \frac{\lambda}{s} \sum_{j=1}^{1298} \frac{1}{1 - \frac{s^2}{\gamma_j^2}} \left\{ G_{1,j} \left[-\frac{s^2}{\gamma_j^2} \cos \gamma_j t \right. \right. \\
& \quad \left. \left. - \frac{s}{\gamma_j} \sin (\gamma_j o) \sin st + \frac{s^2}{\gamma_j^2} \cos (\gamma_j o) \cos st \right] \right. \\
& \quad + G_{2,j} \left[-\frac{s}{\gamma_j} \sin \gamma_j t + \frac{s}{\gamma_j} \sin (\gamma_j o) \cos st \right. \\
& \quad \left. \left. + \frac{s^2}{\gamma_j^2} \cos (\gamma_j o) \sin st \right] \right\}
\end{aligned}$$

$$\begin{aligned}
h_2(t) = \frac{\lambda}{s} \sum_{j=1}^{1298} \frac{1}{1 - \frac{s^2}{\gamma_j^2}} & \left\{ G_{2,j} \left[-\frac{s^2}{\gamma_j^2} \cos \gamma_j t \right. \right. \\
& \left. \left. - \frac{s}{\gamma_j} \sin (\gamma_j o) \sin st + \frac{s^2}{\gamma_j^2} \cos (\gamma_j o) \cos st \right] \right. \\
& + G_{1,j} \left[\frac{s}{\gamma_j} \sin \gamma_j t - \frac{s}{\gamma_j} \sin (\gamma_j o) \cos st \right. \\
& \left. \left. - \frac{s^2}{\gamma_j^2} \cos (\gamma_j o) \sin st \right] \right\}
\end{aligned}$$

The cosine notation again is used to rewrite h_1 and h_2 .

$$h_1(t) = \sum_{j=1}^{2598} N_{1,j} \cos \eta_j t \quad (21)$$

$$h_2(t) = \sum_{j=1}^{2598} N_{2,j} \cos \eta_j t \quad (22)$$

with

$$\begin{aligned}
\cos \eta_j t &= \cos \gamma_j t & j &= 1, 2, \dots, 1298 \\
\cos (\eta_{1298+j}) t &= \sin \gamma_j t \\
\cos (\eta_{2597}) t &= \cos st \\
\cos (\eta_{2598}) t &= \sin st
\end{aligned}$$

$$N_{1,j} = -\frac{\lambda s}{\gamma_j} \left(\frac{1}{1 - \frac{s^2}{\gamma_j^2}} \right) G_{1,j}$$

$$N_{1,1298+j} = -\frac{\lambda}{\gamma_j} \left(\frac{1}{1 - \frac{s^2}{\gamma_j^2}} \right) G_{2,j}$$

$$N_{1,2597} = \frac{\lambda}{s} \sum_{i=1}^{1298} \frac{1}{1 - \frac{s^2}{\gamma_i^2}} \left[\frac{s^2}{\gamma_i^2} G_{1,i} \cos(\gamma_i o) + \frac{s}{\gamma_i} G_{2,i} \sin(\gamma_i o) \right]$$

$$N_{1,2598} = \frac{\lambda}{s} \sum_{i=1}^{1298} \frac{1}{1 - \frac{s^2}{\gamma_i^2}} \left[-\frac{s}{\gamma_i} G_{1,i} \sin(\gamma_i o) + \frac{s^2}{\gamma_i^2} G_{2,i} \cos(\gamma_i o) \right]$$

$$N_{2,j} = -\frac{\lambda s}{\gamma_j} \left(\frac{1}{1 - \frac{s^2}{\gamma_j^2}} \right) G_{2,j}$$

$$N_{2,1298+j} = \frac{\lambda}{\gamma_j} \left(\frac{1}{1 - \frac{s^2}{\gamma_j^2}} \right) G_{1,j}$$

$$N_{2,2597} = \frac{\lambda}{s} \sum_{i=1}^{1298} \frac{1}{1 - \frac{s^2}{\gamma_i^2}} \left[\frac{s^2}{\gamma_i^2} G_{2,i} \cos(\gamma_i o) - \frac{s}{\gamma_i} G_{1,i} \sin(\gamma_i o) \right]$$

$$N_{2,2598} = -\frac{\lambda}{s} \sum_{i=1}^{1298} \frac{1}{1 - \frac{s^2}{\gamma_i^2}} \left[\frac{s}{\gamma_i} G_{2,i} \sin(\gamma_i o) + \frac{s^2}{\gamma_i^2} G_{1,i} \cos(\gamma_i o) \right]$$

A total of 5,000 terms is involved

From the denominators

$$\frac{1}{1 - \frac{s^2}{2\gamma_j}}$$

and the formation structure of ϵ_j , it can be seen that the most important contributions occur when

$$\gamma_i = s \pm p$$

$$\gamma_i = s \pm 2p$$

where p is the orbit-precession rate. As seen in space, the terms containing the corresponding frequencies p and $2p$ contribute most to the spin-axis wander.

Numerical Results

Two examples were investigated, one with the sun line as the vernal equinox, making $\phi = 0$ and $\theta = \pi/2$ in $A(o)$ and figure 1, and the other with the conditions three months later, so that $\phi = \pi/2$ and $\theta = \pi/2 - \epsilon$ (where ϵ is the obliquity of the ecliptic). Both special examples required considerably less algebraic calculations than the general case. These calculations were performed for the most part with a slide rule. MORL constants used for the calculations are

$$\left. \begin{aligned} I_1 &= 4.0 \times 10^5 \text{ slug ft}^2 \\ I_2 &= 6.84 \times 10^7 \text{ slug ft}^2 \\ I_3 &= 6.85 \times 10^7 \text{ slug ft}^2 \end{aligned} \right\} \text{moments of inertia}$$

$$i = 28.6^\circ \quad \text{orbit inclination}$$

$$s = 3.96 \times 10^{-1} \text{ rad/sec} \quad \text{spin}$$

$$p = 1.454 \times 10^{-6} \text{ rad/sec} \quad \text{orbit precession}$$

$$q = 2.34 \times 10^{-4} \text{ rad/sec} \quad \text{orbit angular rate}$$

and

$$\epsilon = 23.5^\circ \quad \text{obliquity of the ecliptic}$$

For each example, an upper bound of 2.5° was found on the angle of wander. The spin axis (maximum moment of inertia) deviates from the instantaneous angular momentum vector by at most 0.4 seconds of arc; hence, there is no coning. Thus, the motion in space is that of a slow excursion at orbit-precession frequency and does not exceed 2.5° .

CONCLUSION

The spin-axis stability (in inertial space) for an asymmetric body under the action of gravity gradient while in orbit about an oblate Earth was investigated. MORL parameters were used. A new method of iterations for rigid motion, resulting from a different technique of expressing attitude, yielded a series of approximations with very fast convergence. It was found that there is no appreciable coning; specifically, the spin axis (axis of maximum moment of inertia) deviates no more than 0.4 seconds of arc from the angular momentum vector as it wanders in space. The slow variation of angular momentum from initial position was found to be tolerable; it amounted to no more than 2.5° . The most important discovery was that the upper bound of 2.5° is independent of boundary conditions which involve different spin-axis angles to the Earth's equator.

APPENDIX

Extension to Elliptic Orbits

The spin-axis stability analysis method as it is applied to an oblate Earth field involves an inexact description of the orbit radius vector. An actual near-circular orbit around an oblate Earth is replaced by uniform circular motion in a plane that precesses at a constant rate about a fixed line, the polar axis. Near-elliptical orbits can be treated similarly by replacing the actual orbit by a truly elliptical orbit in a plane assumed to precess at a constant rate about the polar axis. A second uniform motion representing the precession of perigee also can be added vectorially. Since the last two precessions are simple to incorporate, it will be assumed that the orbit is taken in the \hat{e}_1^* , \hat{e}_2^* plane. Equation (4) shows that the variables of interest in the torque expression are $\frac{1}{R^3(t)}$ and $\hat{r}^*(t)$; therefore these must be obtained as a function of time. To employ the expansion in a small parameter, however, the parameter must be changed to λ' , defined as follows:

$$\lambda' = \frac{3G}{A^3}$$

where A is the semimajor axis. No real change in parameter order of magnitude occurs, hence $\left[\frac{A}{R(t)} \right]^3$ is now of interest

First, $R(t)$ (whose expansion in terms of the eccentricity is given on page 171 of reference 1) must be calculated. By rearranging terms, $R(t)$ can be expressed as Fourier cosine series in which n is the mean angular motion and e is the eccentricity.

$$\begin{aligned} \frac{R(t)}{A} = & 1 + \frac{e^2}{2} + \left(-e + \frac{3}{8}e^3 - \frac{5}{192}e^5 + \dots \right) \cos nt \\ & + \left(-\frac{1}{2}e^2 + \frac{1}{3}e^4 - \frac{1}{16}e^6 + \dots \right) \cos 2nt \end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{3}{8} e^3 + \frac{45}{128} e^5 - \dots \right) \cos 3nt \\
& + \left(-\frac{1}{3} e^4 + \frac{2}{5} e^6 - \dots \right) \cos 4nt \\
& + \left(-\frac{125}{384} e^5 + \dots \right) \cos 5nt \\
& + \left(-\frac{27}{80} e^6 + \dots \right) \cos 6nt \\
& + \dots
\end{aligned}$$

The coefficient of $\cos knt$ is an infinite series in e with the lowest power equal to k . The order of this coefficient is identified as k ; the coefficient is identified as b_k . If a certain e has been chosen, the series can be cut off at some $k = \ell_0$. At a given k , b_k can be carried out to the required accuracy, where it will be called b'_k . Thus,

$$\frac{R(t)}{A} \doteq 1 + \left(\frac{e^2}{2} + \sum_{k=1}^{\ell_0} b'_k \cos knt \right)$$

For $\frac{1}{R^3(t)}$, the Binomial theorem is used to group the terms as follows:

$$\begin{aligned}
\left[\frac{A}{R(t)} \right]^3 &= 1 - 3 \left(\frac{e^2}{2} + \sum_{k=1}^{\ell_0} b'_k \cos knt \right) \\
&+ 6 \left(\frac{e^2}{2} + \sum_{k=1}^{\ell_0} b'_k \cos knt \right)^2 \\
&- 10 \left(\frac{e^2}{2} + \sum_{k=1}^{\ell_0} b'_k \cos knt \right)^3 \\
&+ \dots
\end{aligned}$$

and convergence is assured because $\frac{R(t)}{A} = 1 + \xi$ with $|\xi| < 1$ for all e . Note that only a finite number of terms significantly contributes because of the products of the b'_k . In particular, the j^{th} power of a bracket will contain coefficients of the smallest order j . In other words, the desired functional relation is a finite sum of finite products of circular terms.

Next, it is important to express the time variation of $\hat{\mathbf{r}}^*(t)$, which is best done by giving the 1 and 2 components of this unit vector in terms of the true anomaly ν .

$$\hat{\mathbf{r}}^*(t) = \begin{pmatrix} \cos \nu \\ \sin \nu \\ 0 \end{pmatrix}$$

An expansion of ν in terms of e from page 171 of reference 1 can be arranged to form a modified Fourier sine series.

$$\begin{aligned} \nu = nt &+ \left(2e - \frac{3}{12} e^3 + \frac{50}{960} e^5 - \dots \right) \sin nt \\ &+ \left(\frac{5}{4} e^2 - \frac{44}{96} e^4 + \frac{85}{960} e^6 - \dots \right) \sin 2nt \\ &+ \left(\frac{13}{12} e^3 - \frac{645}{960} e^5 + \dots \right) \sin 3nt \\ &+ \left(\frac{103}{96} e^4 - \frac{902}{960} e^6 + \dots \right) \sin 4nt \\ &+ \left(\frac{1097}{960} e^5 - \dots \right) \sin 5nt \\ &+ \left(\frac{1223}{960} e^6 - \dots \right) \sin 6nt \\ &+ \dots \end{aligned}$$

As before, the coefficient (series in e) a'_k of $\sin knt$ is of order k , hence the Fourier series may be cut off at $k = l_1$ for a chosen e . It is again assumed each a'_k is carried out to the required accuracy, a'_k .

$$\cos \left(nt + \sum_{k=1}^{l_1} a'_k \sin knt \right)$$

$$\sin \left(nt + \sum_{k=1}^I a'_k \sin knt \right)$$

If the above expressions are expanded by the angle sum rule, additive terms of products of the following form appear:

$$\cos (a'_k \sin knt)$$

$$\sin (a'_k \sin knt)$$

These expressions can be written as infinite sums of integral-order Bessel functions (reference 2, page 361).

$$\cos (a'_k \sin knt) = J_0(a'_k) + 2 \sum_{i=1}^{\infty} J_{2i}(a'_k) \cos \left[2i knt \right]$$

$$\sin (a'_k \sin knt) = 2 \sum_{i=0}^{\infty} J_{2i+1}(a'_k) \sin \left[(2i+1) knt \right]$$

In the right-hand sides, only a finite number of terms contributes because of the smallness of the a'_k for large enough k and the form of J_i .

$$J_i(x) = \left(\frac{1}{2} x \right)^i \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{4} x^2 \right)^j}{j! \Gamma(i+j+1)}$$

Thus, both $\left[\frac{A}{R(t)} \right]^3$ and the components of $\hat{r}^*(t)$ are expressed as finite sums of finite products of circular functions of time. By the operations of equations (11) and (12) and the formation of $\omega_1(t)$ in equation (8), this

condition is retained. (Note that in equation (12) the factor $\left[\frac{A}{R(t)} \right]^3$ must be inserted.)

It has been shown that algebraic manipulations of circular functions can be performed by a computer. Hence, a very accurate description of elliptic motion is integrable for gravity torque effects on an asymmetric body. The details and the amount of work depend on the value of e .

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